Exercise 2

Consider the equations in Exercise 1 supplemented by the initial data

$$u(x,0) = f(x), \quad v(x,0) = h(x).$$

(a) Show that the appropriate initial data for the wave equation for u is

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = -h'(x).$$

(b) Find the appropriate initial data for the wave equation for v.

Solution

The two given conditions in Exercise 1 are

$$\frac{\partial u}{\partial t} = -\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial t} = -\frac{\partial u}{\partial x}.$$

Since u and v satisfy the wave equation, which is second-order in t, two initial conditions are needed for there to be a well-defined solution. The initial value problem for u is as follows.

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \ -\infty < t < \infty \\
u(x,0) &= f(x) \\
\frac{\partial u}{\partial t}(x,0) &= -\frac{\partial v}{\partial x}(x,0) \\
&= -\frac{d}{dx}[v(x,0)] \\
&= -\frac{d}{dx}[h(x)] \\
&= -h'(x)
\end{aligned}$$

The initial value problem for v is very similar.

$$\begin{aligned}
\frac{\partial^2 v}{\partial t^2} &= \frac{\partial^2 v}{\partial x^2}, \quad -\infty < x < \infty, \ -\infty < t < \infty \\
v(x,0) &= h(x) \\
\frac{\partial v}{\partial t}(x,0) &= -\frac{\partial u}{\partial x}(x,0) \\
&= -\frac{d}{dx}[u(x,0)] \\
&= -\frac{d}{dx}[f(x)] \\
&= -f'(x)
\end{aligned}$$